Convergence of unmatched ray-driven forward and pixel-driven backprojection in tomography

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Computational Uncertainty Quantification for Inverse problems

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DTU Compute
Department of Applied Mathematics and Computer Science

Tomography in a nutshell

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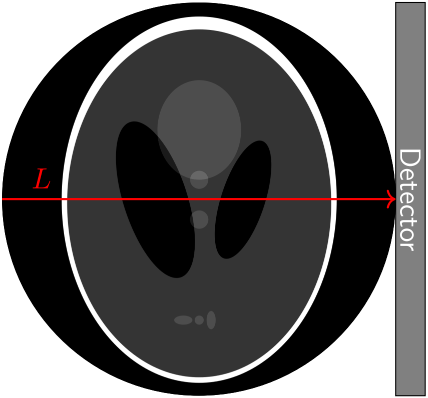

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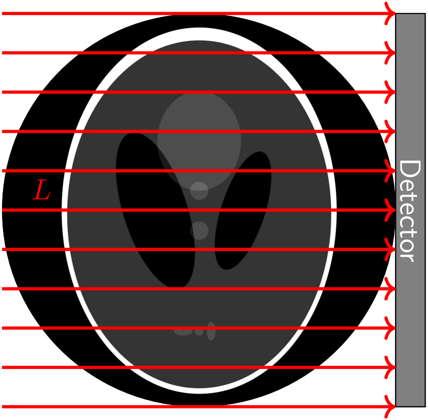

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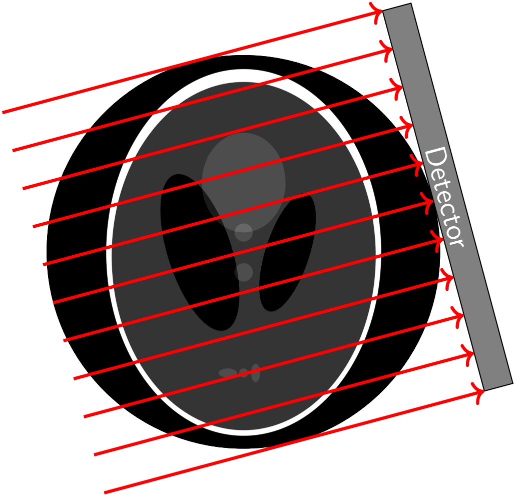

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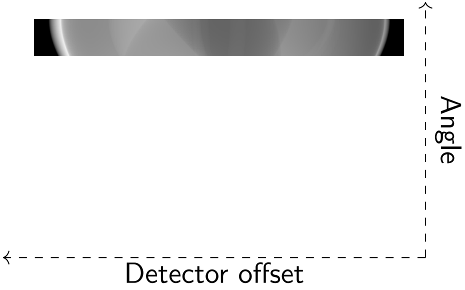

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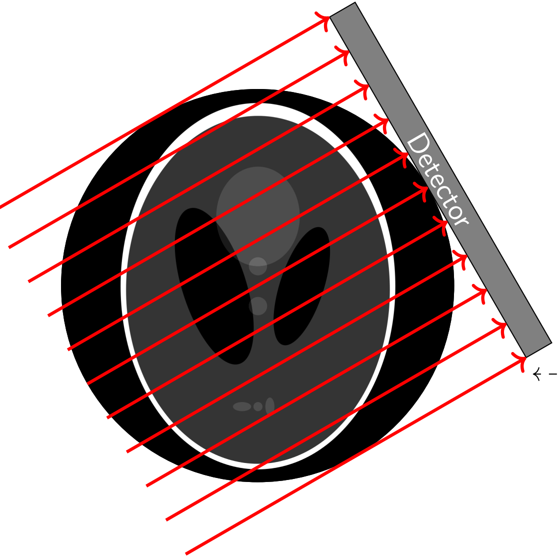

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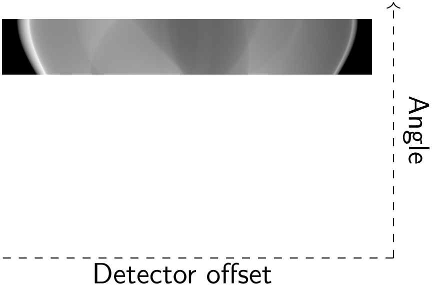

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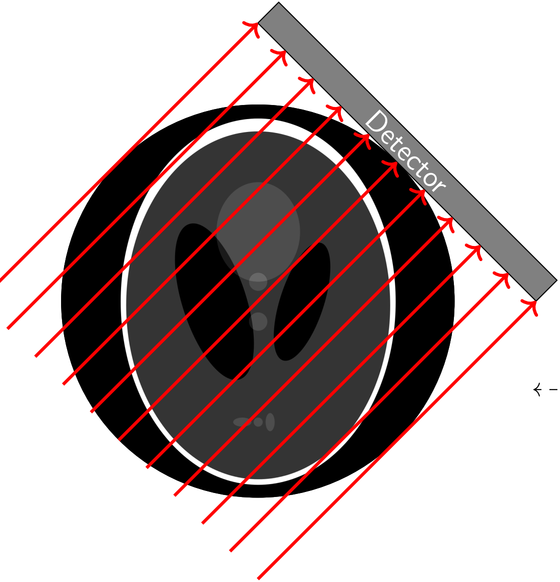

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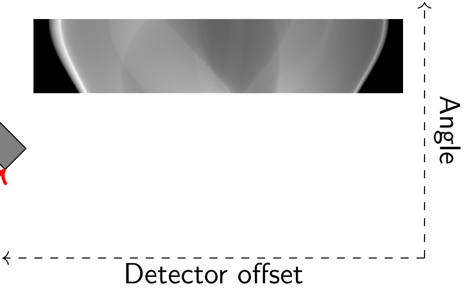

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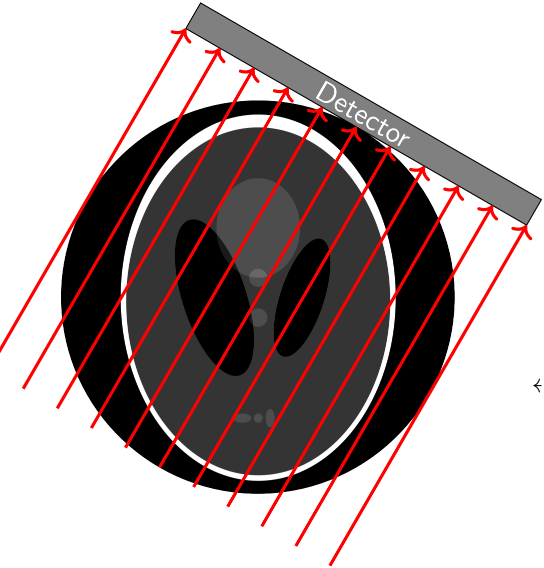

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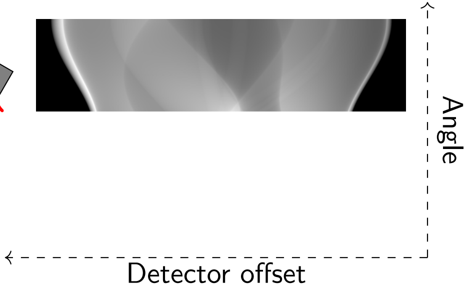

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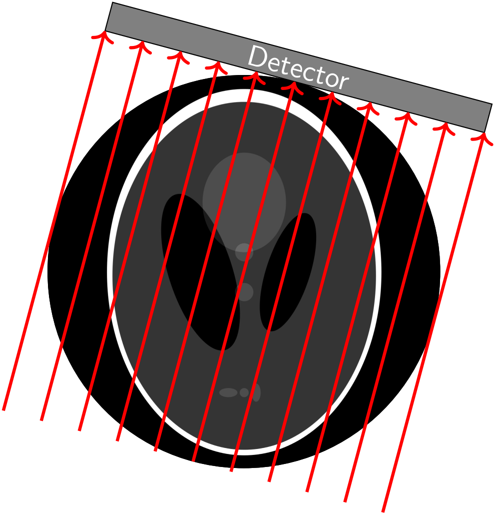

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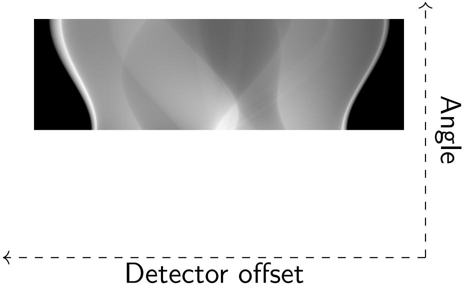

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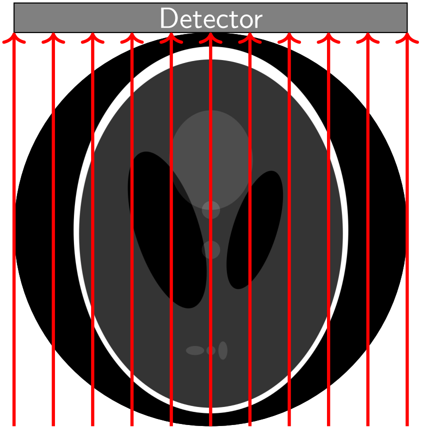

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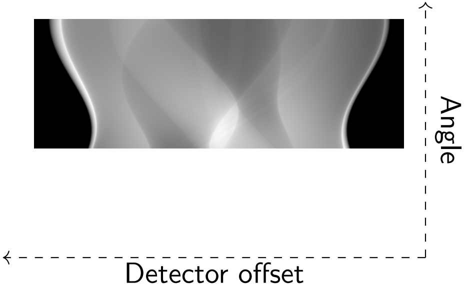

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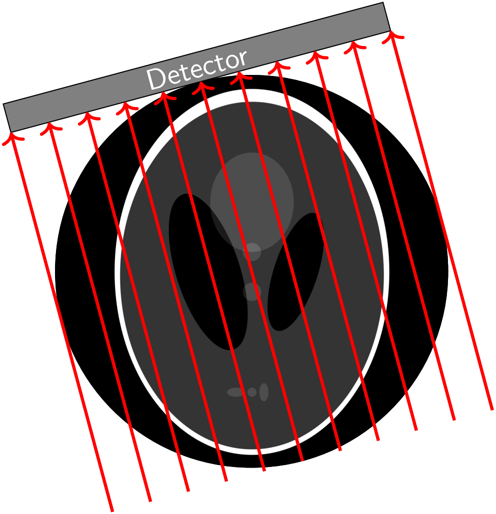

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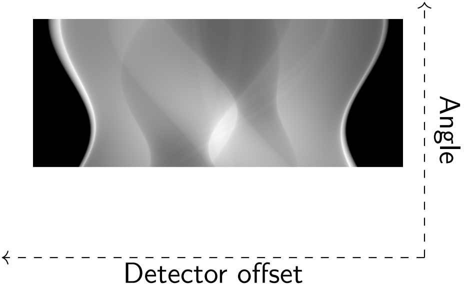

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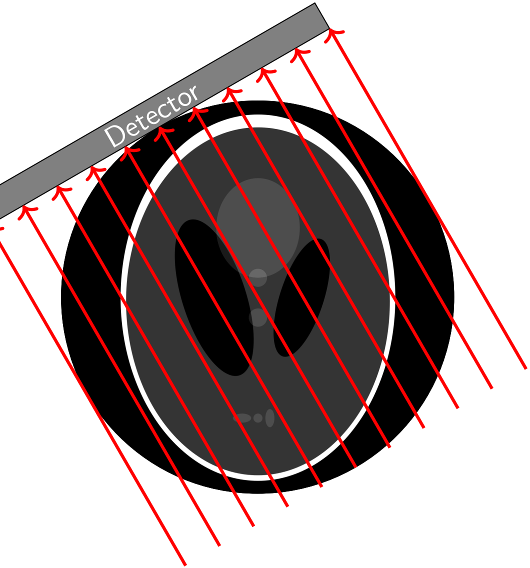

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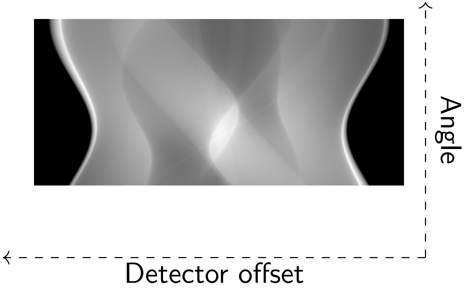

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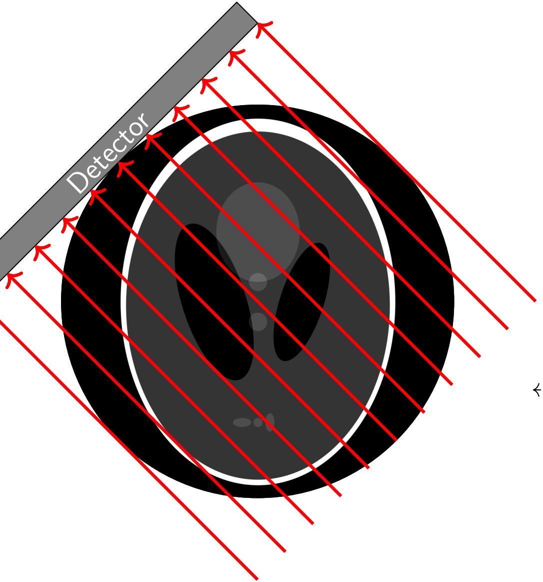

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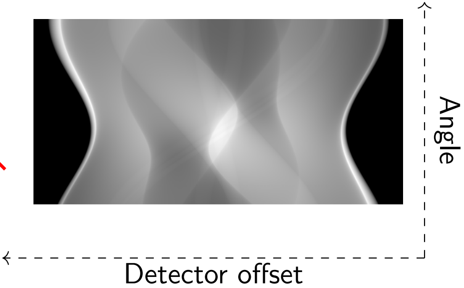

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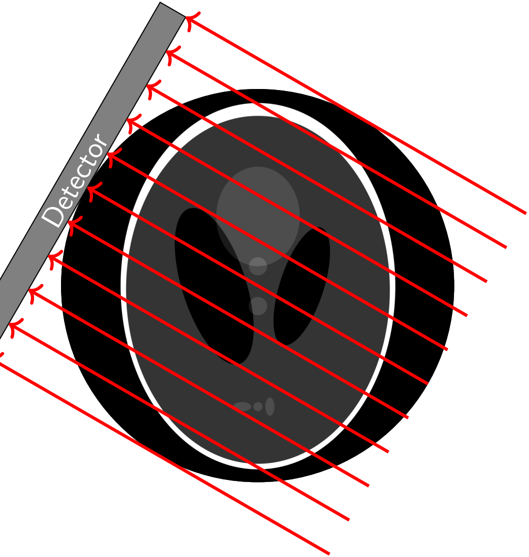

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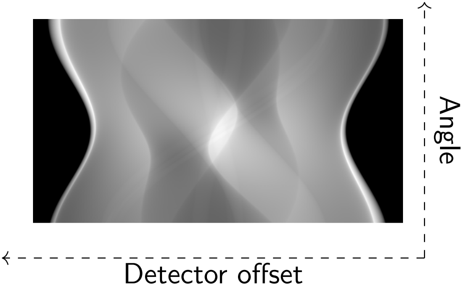

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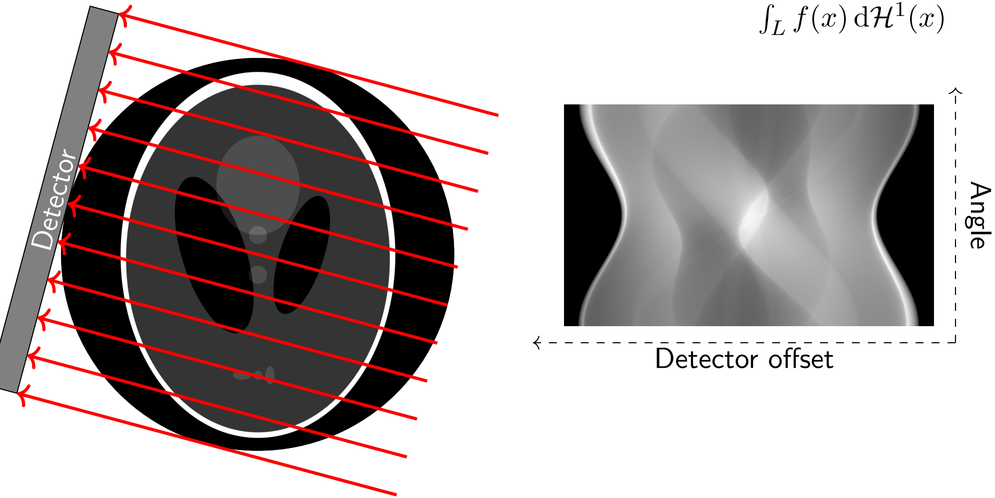

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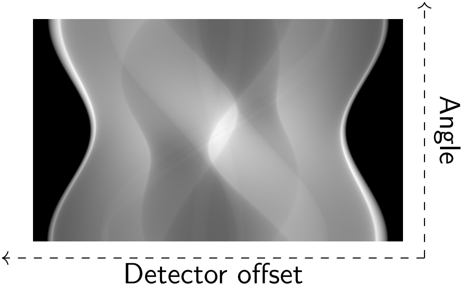

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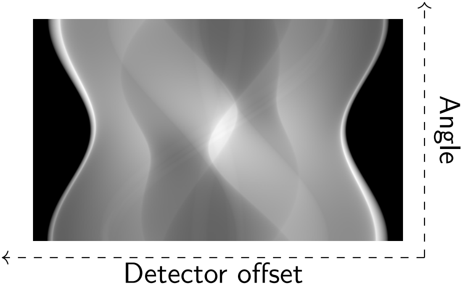

- ${\color{red}g}$ ... loss of energy

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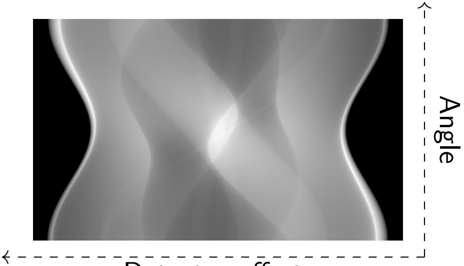

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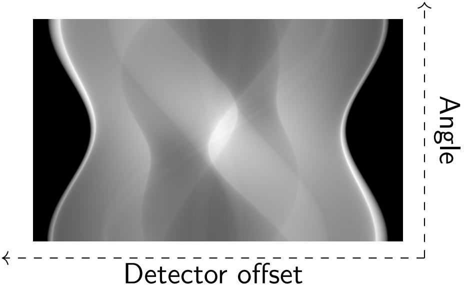

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- $\mathcal{R}$ ... Radon transform

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Goal: Investigate approximation properties!

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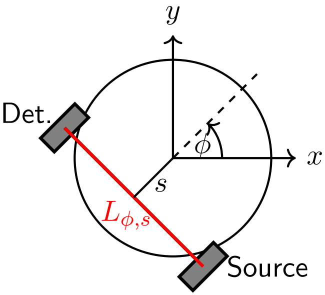

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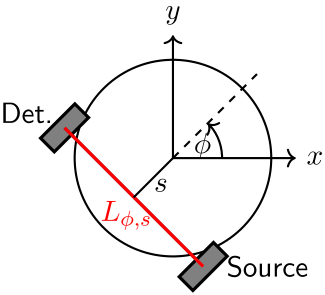

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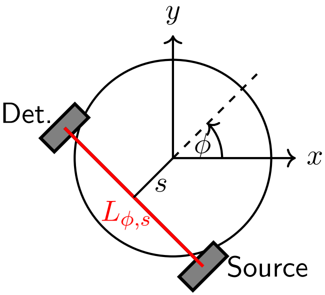

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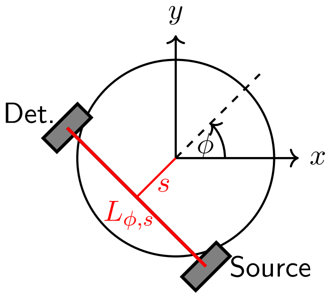

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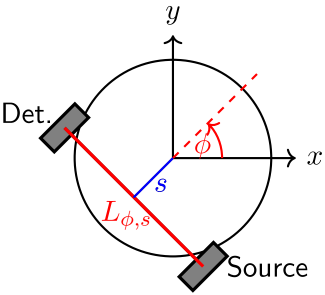

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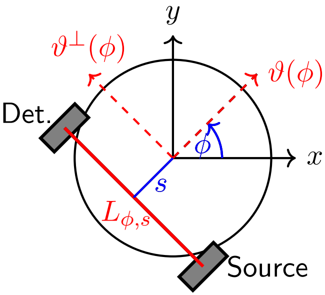

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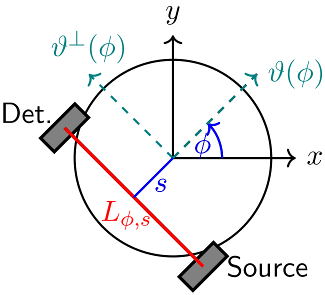

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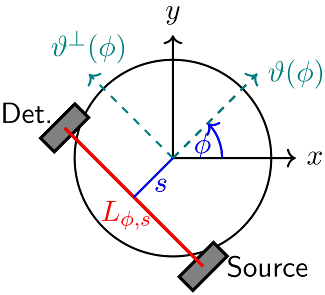

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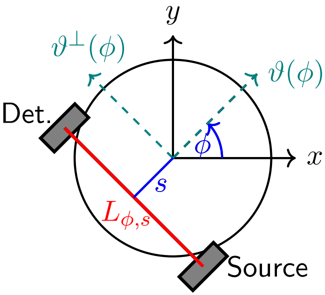

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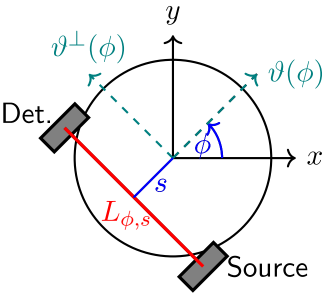

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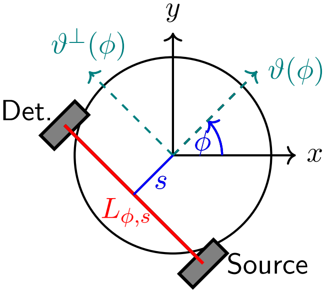

Theorem

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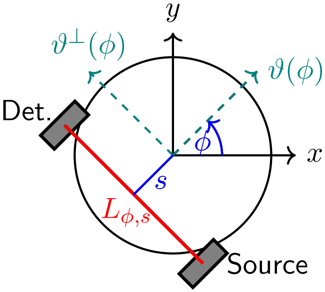

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The Radon transform $\mathcal{R}\colon L^2(\Omega) \to L^2(\mathcal{S})$ is a continuous linear operator.

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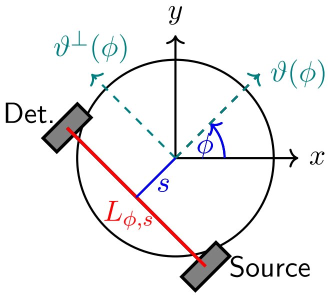

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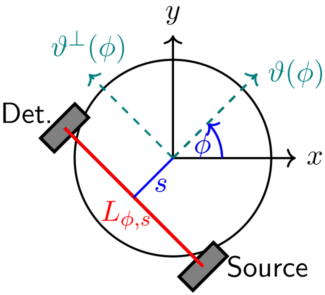

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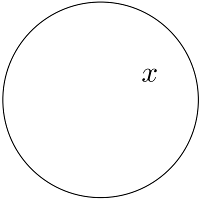

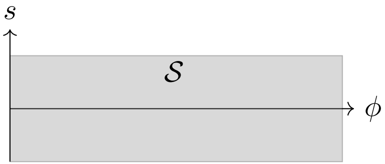

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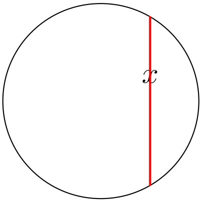

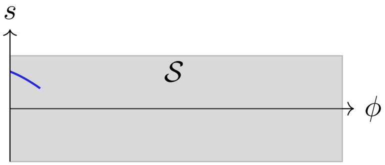

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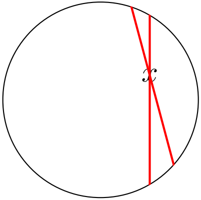

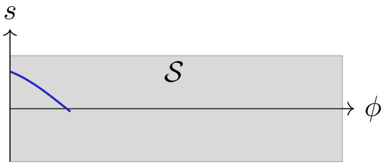

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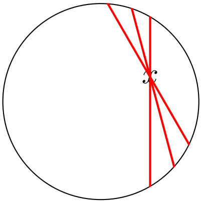

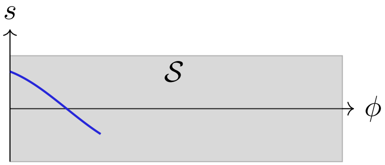

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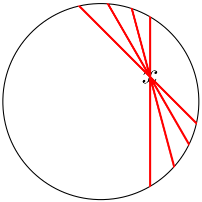

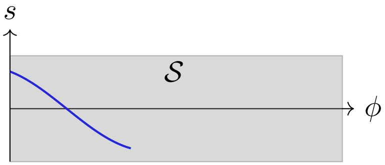

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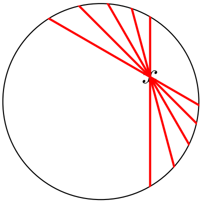

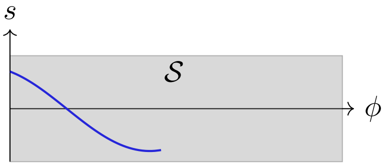

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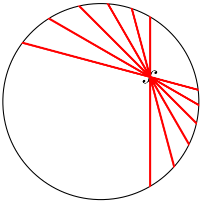

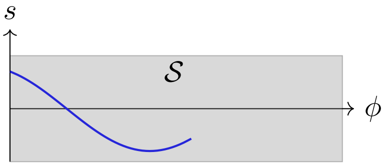

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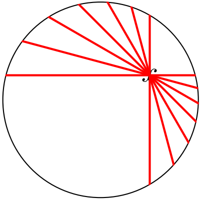

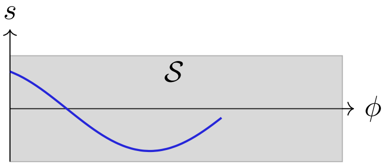

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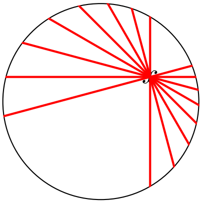

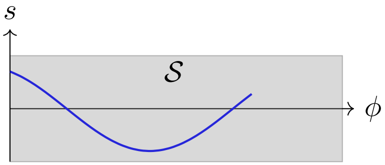

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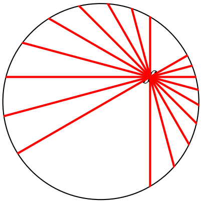

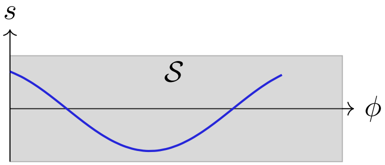

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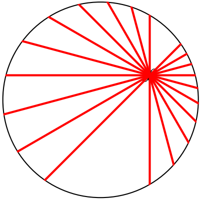

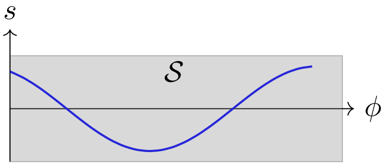

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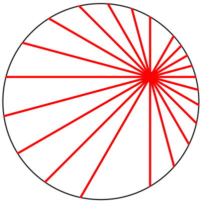

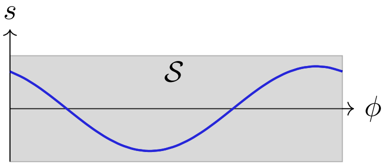

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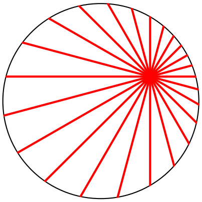

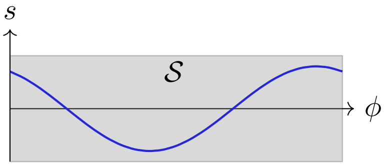

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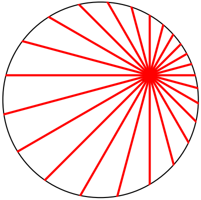
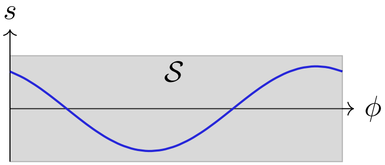
The operator $\mathcal{R}^*$ is indeed the adjoint to $\mathcal{R}\colon L^2(\Omega) \to L^2(\mathcal{S})$.

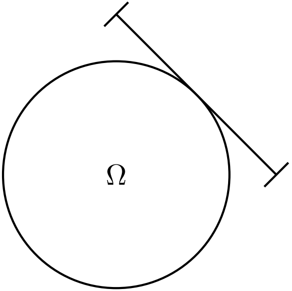


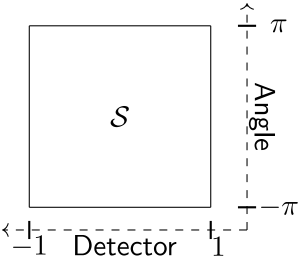


Outline

- The Radon Transform
- Convolutional Discretization Schemes
- Convergence Results
- Numerical Experiments
- The L^2 Optimal Discretization

Parallel beam setting





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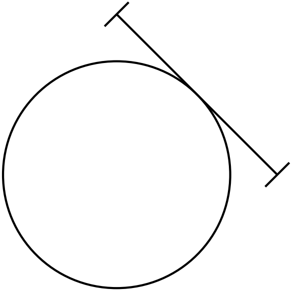

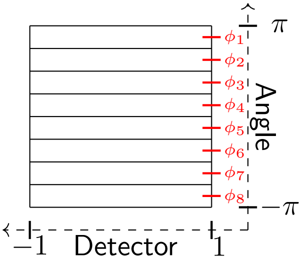

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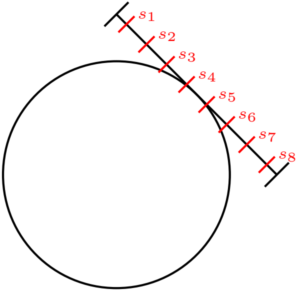

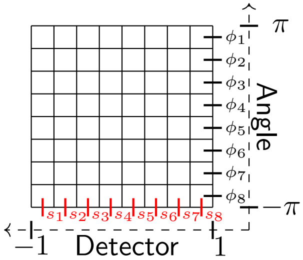

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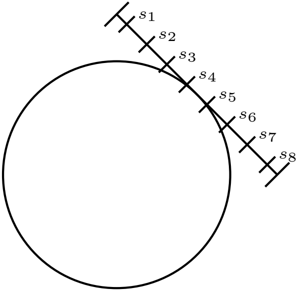

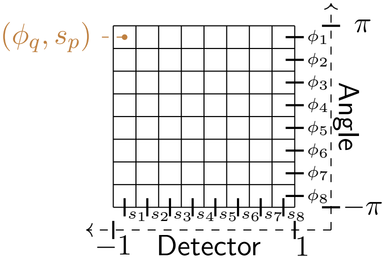

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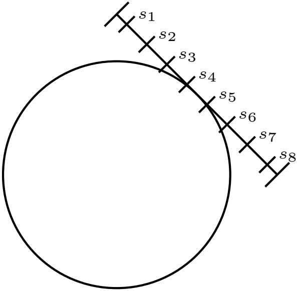

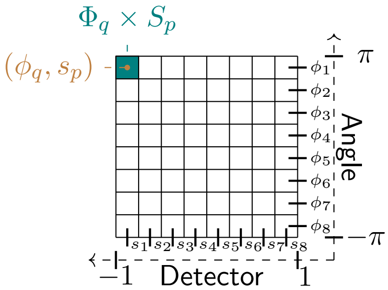

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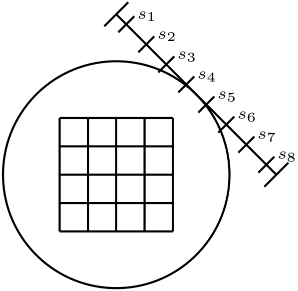

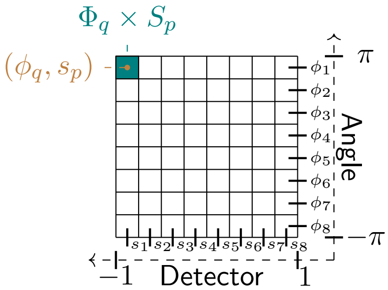

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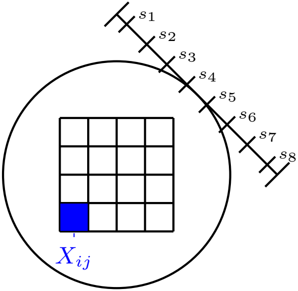

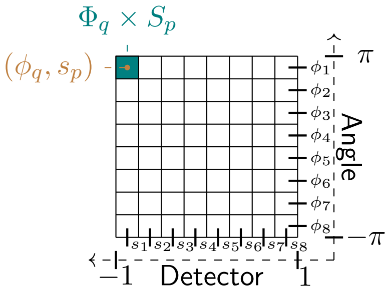

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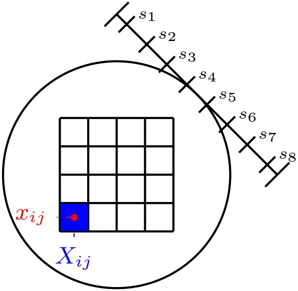

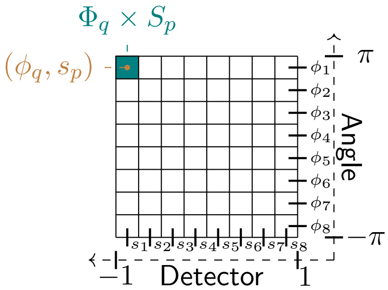

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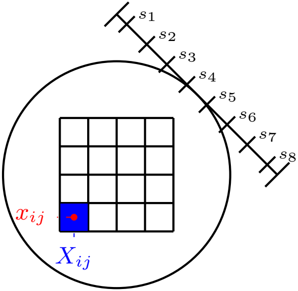
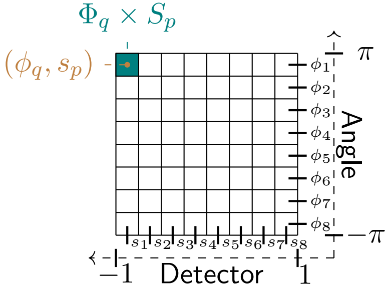
- Detector offsets: Equispaced $s_0 < \cdots < s_{N_s - 1}$ with corresponding pixels $S_p = s_p + [-\frac{\delta_s}{2}, \frac{\delta_s}{2}[$.
- Image pixels: Equispaced pixels X_{ij} with centers x_{ij} and width δ_x.
- Discretization parameters: δ_s, δ_ϕ and δ_x.





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- Note $\mathcal{R}_\delta^\omega\colon L^2(\Omega)\to L^2(\mathcal{S})$ and $\mathcal{R}_\delta^\omega\colon L^2(\mathcal{S})\to L^2(\Omega)$ finite rank.

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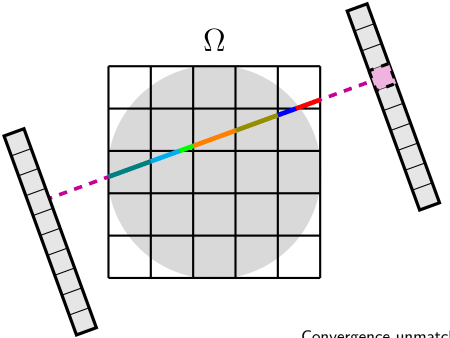

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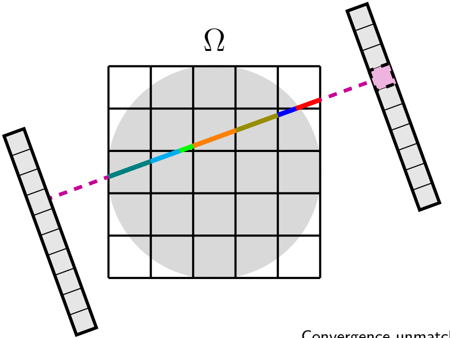

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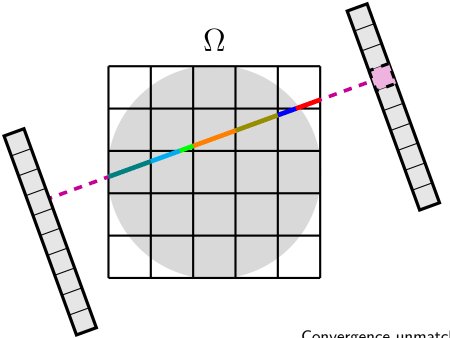

Ray-driven weights

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Given δ and $\phi \in [0, \pi[$, we set $\overline{s}(\phi) := \frac{\delta_x}{2}(|\cos(\phi)| + |\sin(\phi)|)$, $\underline{s}(\phi) := \frac{\delta_x}{2}(\big||\cos(\phi)| - |\sin(\phi)|\big|)$ and $\kappa(\phi) := \min\left\{\frac{1}{|\cos(\phi)|}, \frac{1}{|\sin(\phi)|}\right\}$. We define the ray-driven weight function for $t \in \mathbb{R}$ according to

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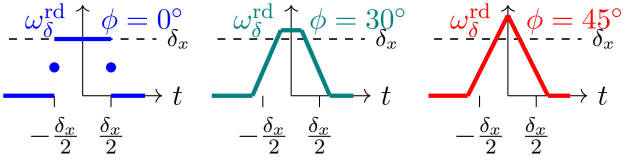

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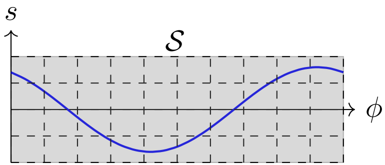
$$[\mathcal{R}^* g](x_{ij}) = \int_{[-\pi,\pi[} g(\phi, x_{ij} \cdot \vartheta(\phi)) \, \mathrm{d}\phi$$



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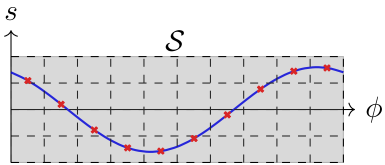

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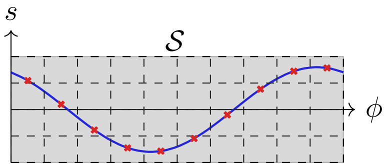

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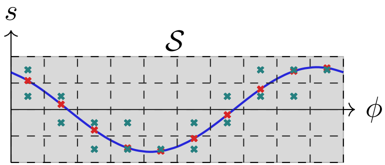

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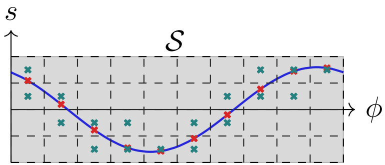

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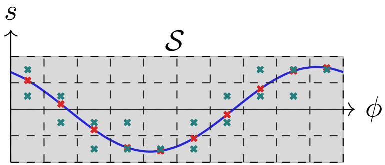

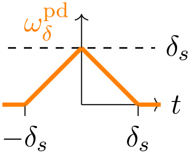

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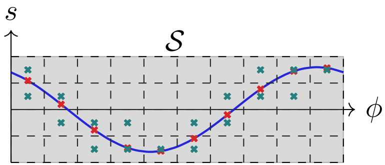

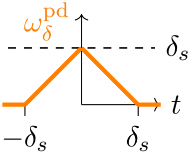

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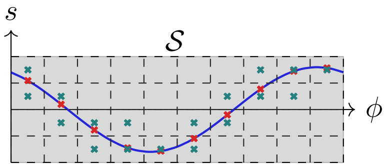

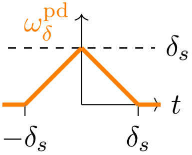

Outline

- The Radon Transform
- Convolutional Discretization Schemes
- **Convergence Results**
- Numerical Experiments
- The L^2 Optimal Discretization

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$\square$

Outline

Example 1

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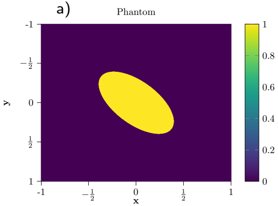

Example 1

$f = \chi_E$ with $[\mathcal{R}f](\phi, s) = \eta(\phi)\sqrt{1-(\xi(\phi)s)^2}$.

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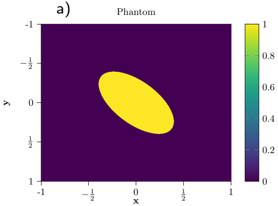

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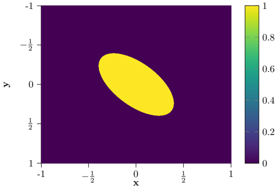

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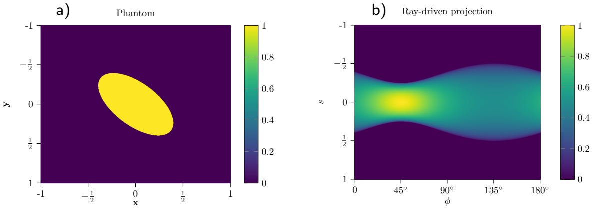

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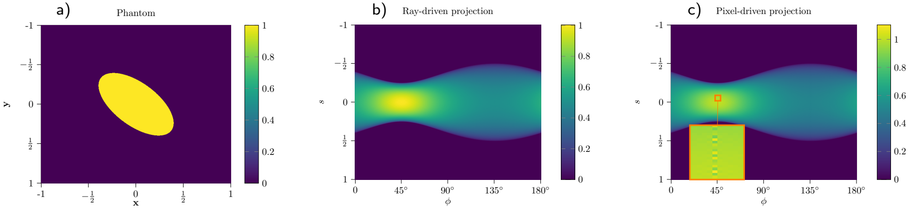

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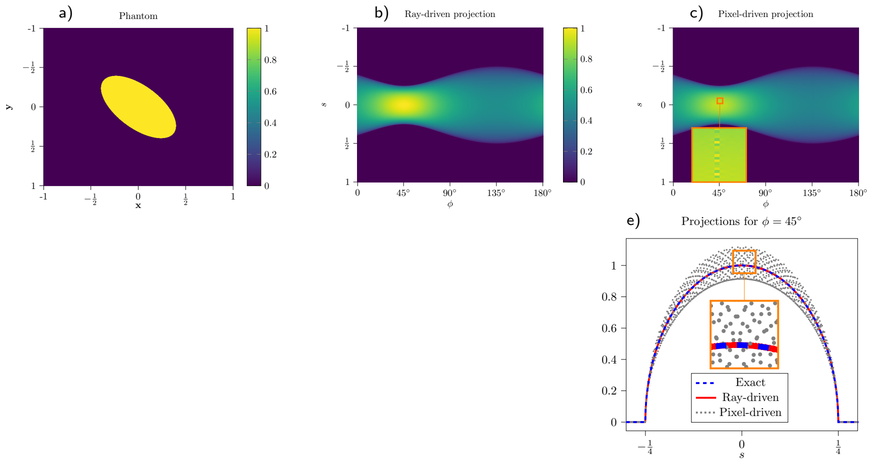

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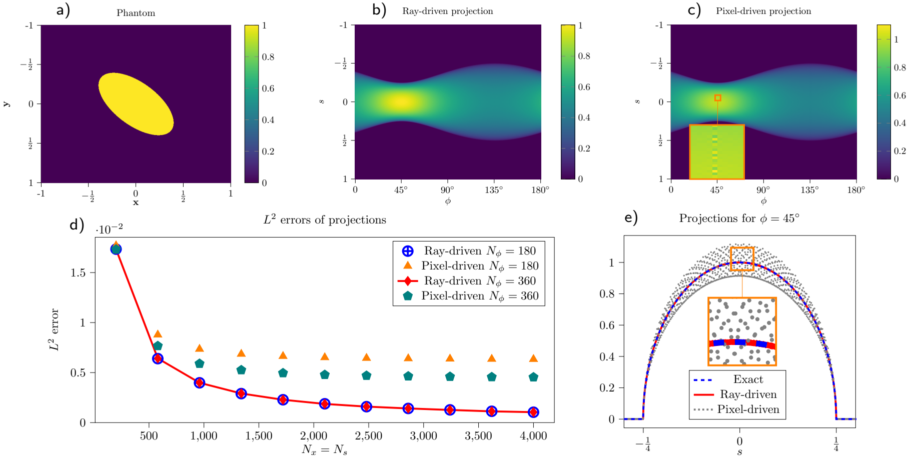

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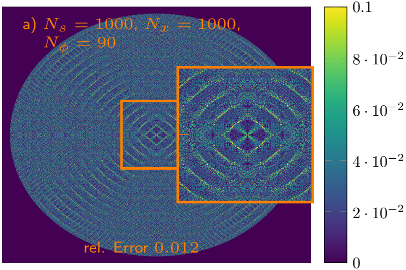

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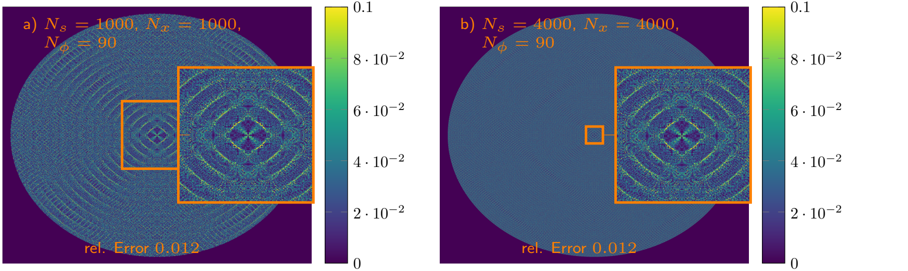

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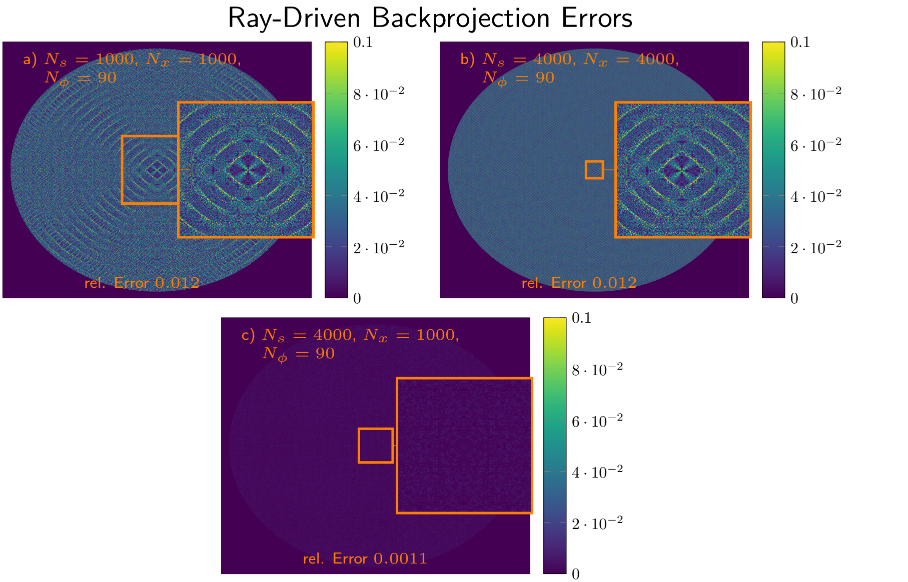

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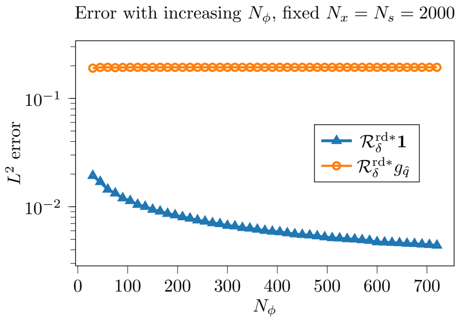

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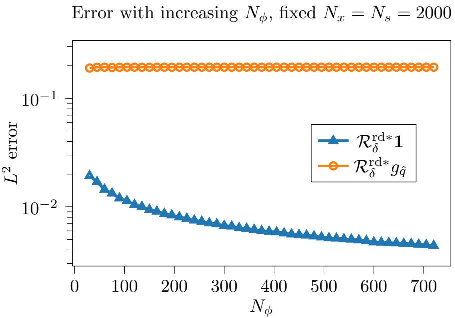

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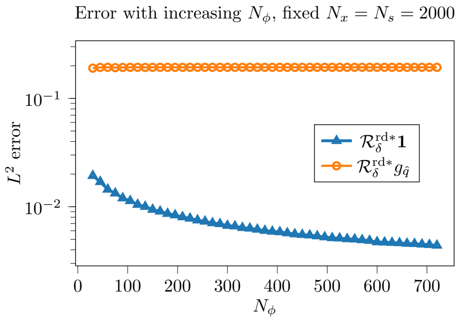

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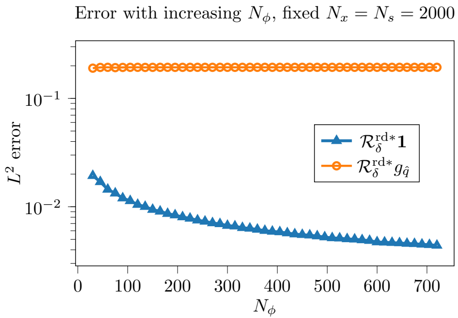

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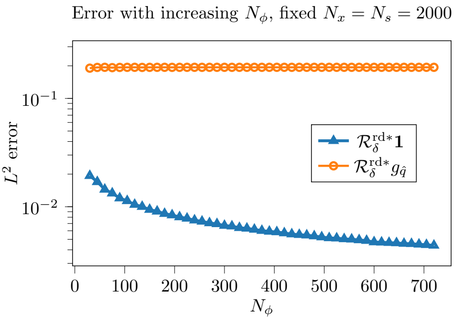

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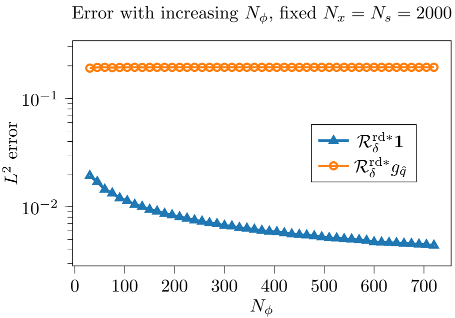

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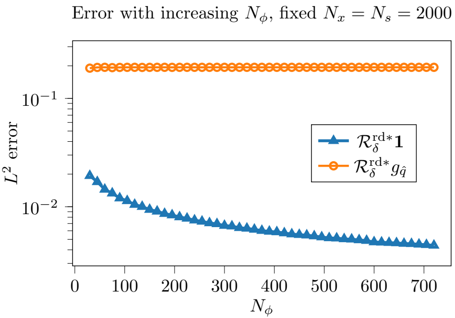

Outline

- The Radon Transform
- Convolutional Discretization Schemes
- Convergence Results
- Numerical Experiments
- The L^2 Optimal Discretization

A generic finite rank approximation

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L^2(\Omega) \arrow[r, "\mathcal{R}"] \arrow[d, "\mathcal{P}_U"'] \arrow[dr, phantom, "\mathcal{R}_\delta"] & L^2(\mathcal{S}) \\
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- Other choices possible, e.g., piecewise linear.

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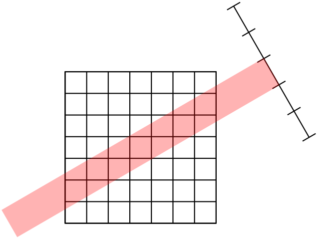

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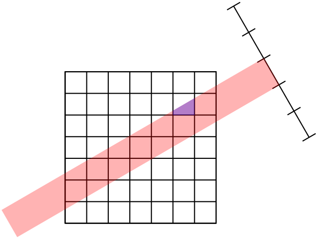

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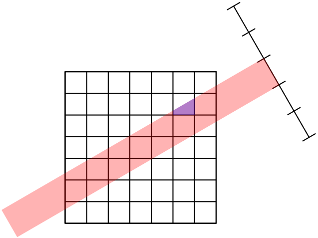

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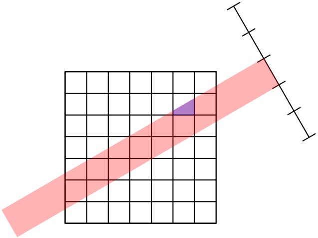

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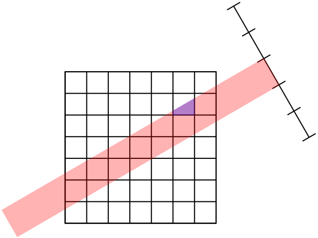

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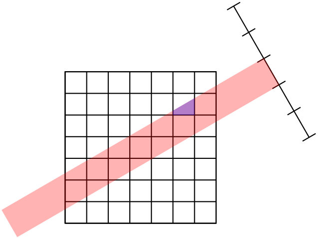

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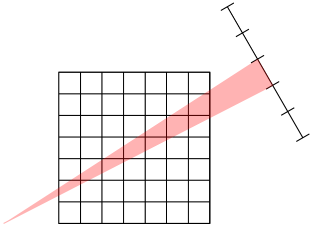

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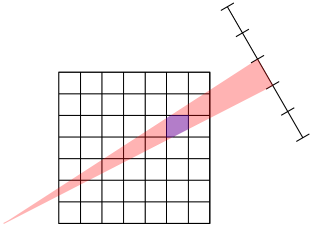

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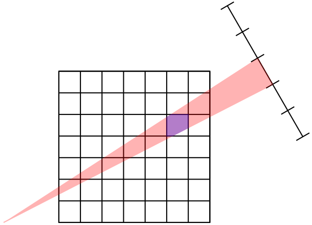

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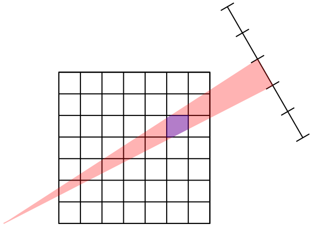

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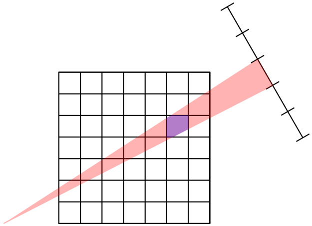

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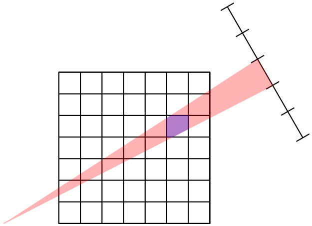

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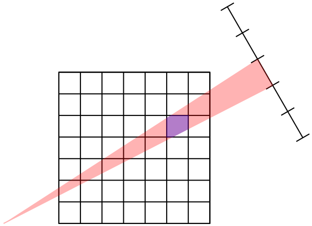

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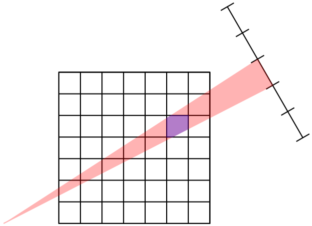

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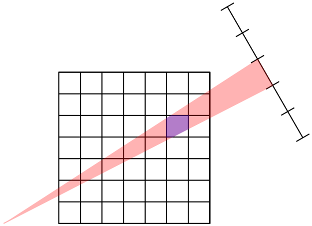

Weighted Strip models

- The described weights correspond to the 'Strip Model'.
- Due to the orthogonality we now this is optimal discretization.
- Is naturally an matched operator pair.
- Extendable to fanbeam and conebeam geometries.
  - Creating novel Weighted Strip Models.
  - Multiplicative factor related to Jakobi-determinants reflecting geometry.
  - Novel-discretizations that are (in some sense) better than existing ones.
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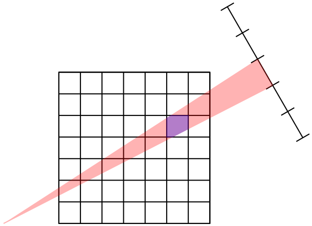

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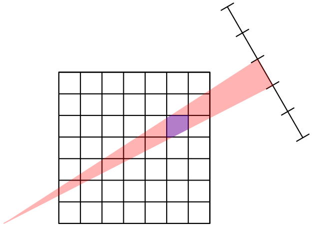

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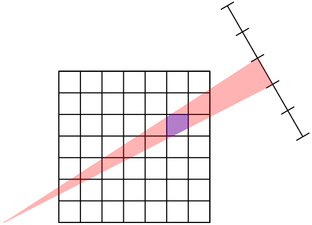

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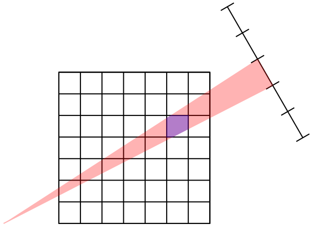

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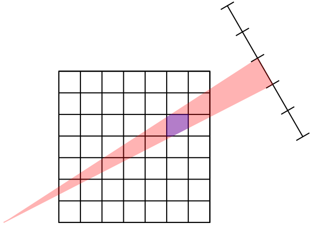

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- Outlook:
    - Extensions to Fanbeam and Conebeam transformations,
    - Implementation and testing of optimal discretizations,
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References

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