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Basic information: Spectral theory

For X a Banach space and an operator $T\colon X \to X$ linear and continuous (we say $T \in L(X)$), the resolvent set is defined as

$$\rho(T) = \big\{\lambda \in \mathbb{K} \mid (\lambda - T) \text{ is bijective}\big\}. \tag{1}$$

The set $\sigma(T) := \mathbb{K} \setminus \rho(T)$ is called the spectrum of T and can further be partitioned into the point spectrum σ_p, the continuous spectrum σ_c, and the residual spectrum σ_r:

$$\sigma_p(T) = \big\{\lambda \in \sigma(T) \mid (\lambda - T) \text{ is not injective}\big\}, \tag{2}$$
$$\sigma_c(T) = \big\{\lambda \in (\sigma(T) \setminus \sigma_p(T)) \mid (\lambda - T) \text{ has dense range}\big\}, \tag{3}$$
$$\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T)). \tag{4}$$

Note that the spectrum is a compact set contained in $\overline{B(0, \|T\|)}$, and $\lambda \in \sigma(T)$ if and only if $\overline{\lambda} \in \sigma(T^*)$.
On a Hilbert space H, we say $T \in L(H)$ is selfadjoint if $T^* = T$ and normal if $T^*T = TT^*$ (with the identification $H \overset{\wedge}{=} H^*$). For selfadjoint operators $\sigma(T) \subset \overline{W(T)} \subset \mathbb{R}$ with $W(T) := \big\{\langle Tx, x\rangle \bigm| \|x\| = 1\big\}$, and for a normal operator T, x is an eigenvector to T w.r.t. λ implies that x is also eigenvector of T^* w.r.t. $\overline{\lambda}$. For normal T, the eigenspaces $\ker(\lambda - T) \perp \ker(\mu - T)$ for $\lambda, \mu \in \sigma_p(T)$ with $\lambda \neq \mu$, i.e., eigenspaces are orthogonal to each other.
When $K \in L(X)$ is compact (i.e., the image of bounded sets are precompact), $\sigma(K) = \sigma_p(K)$, i.e., $\lambda - K$ not being bijective implies non-injectivity. Moreover, $\sigma(K)$ is countable, bounded and the only accumulation point can be 0. For $\lambda \in \sigma(K) \setminus \{0\}$ the eigenspace $\ker(\lambda - K)$ is finite-dimensional. For $T \in L(H)$ being compact and normal, we denote by E_λ the orthogonal projection onto the eigenspace of λ, and by orthogonality $\|\sum_{\lambda \in \sigma(T)} c(\lambda) E_\lambda x\|^2 = \sum_{\lambda \in \sigma(T)} |c(\lambda)|^2 \|E_\lambda x\|^2$ for any (bounded) function $c\colon \sigma(T) \mapsto \mathbb{K}$. A compact and selfadjoint operator $T \in L(H)$ possesses the representation (eigenvalue decomposition)

$$T = \sum_{\lambda \in \sigma(T)} \lambda E_\lambda. \tag{5}$$

Example 7.1) [Spectrum of shift operators]

We consider the $\mathbb{R}$-Hilbert space $H = l^2(\mathbb{N}, \mathbb{R}) := \big\{(x_n)_{n\in\mathbb{N}} \bigm| x_n \in \mathbb{R}\ \forall n \in \mathbb{N},\ \|x\|_{l^2}^2 := \sum_{n\in\mathbb{N}} |x_n|^2 < \infty\big\}$. We define the left and right shift operators $S_l, S_r\colon H \to H$ according to

$$S_r x = (0, x_1, x_2, \cdots), \qquad \text{and} \qquad S_l x = (x_2, x_3, \dots). \tag{6}$$

It is trivial to show that $S_r = S_l^*$ and $\|S_r\| = 1$.

a) Let $T \in L(X)$ for a Banach space X. Show that $\lambda \in \sigma_r(T)$ implies $\overline{\lambda} \in \sigma_p(T^*)$. Further, show that $\lambda \in \sigma_c(T)$ if and only if $\overline{\lambda} \in \sigma_c(T^*)$.

Also, check whether S_r and S_l are compact and/or normal?

b) Compute $\sigma_p(S_l)$, $\sigma_c(S_l)$, $\sigma_r(S_l)$ and determine the eigenspaces associated with $\lambda \in \sigma_p(S_l)$.

c) Compute $\sigma_p(S_r)$, $\sigma_c(S_r)$, $\sigma_r(S_r)$ and identify the eigenspaces associated with $\lambda \in \sigma_p(S_r)$.

Hint. *For a), recall the connections between* $\mathrm{Rg}(T)$ *and* $\ker(T^*)$*. Problems b) and c) are connected via a) and must be solved simultaneously.*

Remark. *The identification of the different spectrum types can sometimes be quite difficult, in particular showing that the range of an operator is (or is not) dense is often hard (without explicitly constructing a suitable sequence). It can help to consider the spectrum of the adjoint operator to gain further insight.*

Example 7.2) [Approximate eigenvalue]
Let H be a Hilbert space and $T \in L(H)$. We call $\lambda \in \mathbb{K}$ an approximate eigenvalue if there is a sequence of $(x_n)_n$ in H such that $\|x_n\| = 1$ and $\|\lambda x_n - Tx_n\| \to 0$. Show that for normal T, every $\lambda \in \sigma(T)$ is an approximate eigenvalue.

Hint. *For* $\lambda \in \sigma(T)$*, consider the selfadjoint operator* $V = \lambda T^* + \overline{\lambda} T - TT^*$ *and show* $|\lambda|^2 \in \sigma(V)$*.*

Remark. *Sometimes it is convenient to know that something is an approximate eigenvalue, in order to find a sequence with certain properties. Moreover, note that the definition of approximate eigenvalue for* $\lambda \in \sigma(T) \setminus \sigma_p(T)$ *means that the inversion operator* $(\lambda - T)^{-1}$ *cannot be continuous.*

Example 7.3) [Continuous functional calculus for compact operators]
Let H be a Hilbert space and $T \in L(H)$ be compact and selfadjoint. We define (with E_λ as in the 'Basic Information') the operator $\phi\colon \mathcal{C}(\sigma(T)) \to L(H)$ according to

$$\phi(f) = \sum_{\lambda \in \sigma(T)} f(\lambda) E_\lambda. \tag{7}$$

a) Show that ϕ is well defined, linear and that $\|\phi(f)\|_{L(H)} = \|f\|_\infty$ and consequently, ϕ is continuous with respect to the supremum norm on $\mathcal{C}(\sigma(T))$ and the operator norm on $L(H)$. Further, show for $(f_n)_n$ a sequence in $\mathcal{C}(\sigma(T))$ such that $\sup_n |f_n| < \infty$ and $f_n \to f \in \mathcal{C}(\sigma(T))$ pointwise, then also $[\phi(f_n)](x) \to [\phi(f)](x)$ in H for each $x \in X$.

b) Show for $f, g \in \mathcal{C}(\sigma(T))$

$$\phi(t \mapsto 1) = \mathrm{id}_H, \quad \phi(t \mapsto t) = T, \quad \phi(fg) = \phi(f)\phi(g), \quad \phi(\overline{f}) = \phi(f)^*, \tag{8}$$

where fg denotes the pointwise multiplication while $\phi(f)\phi(g)$ denotes composition of linear functionals and $\overline{f}$ refers to pointwise complex conjugation.

Remark. *The function* ϕ *is known as the (continuous) functional calculus (often also used with the notation* $f(T) := \phi(f)$*), which is an important tool in analysis as it allows to apply functions onto linear operators in a reasonable way, in particular transferring many properties from* f *onto* $f(T)$*. Note that one can extend this construction to bounded measurable functions and even unbounded measurable functions (but then also* $\phi(f)$ *is not continuous nor defined everywhere). Moreover, one can extend this concept to bounded selfadjoint operators* T *and even to unbounded selfadjoint operators* T*. Then the sum in (7) must be replaced by an integral (as the spectrum is not necessarily countable) with a vector-valued measure (the so-called spectral measure of* T*).*