Richard Huber
richard.huber@uni-graz.at

Basic information: Approximation via convolution
Given functions $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ the convolution $f * g \in L^r(\mathbb{R}^d)$ with $p, q, r \in [1, \infty]$ such tha $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ is defined according to

$$[f * g](x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y \qquad \text{for almost every } x \in \mathbb{R}^d, \quad (1)$$

where, by the Young inequality, $\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$ holds. Further, $\mathrm{supp}(f * g) \subset \mathrm{supp}(f)+\mathrm{supp}(g)$ and convolution with functions f on a domain Ω is understood as convolution with a zero extension of f, and subsequent restriction to Ω.
We call ϕ a mollifier function if $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $\phi \geq 0$, $\int_{\mathbb{R}^d} \phi\,\mathrm{d}x = 1$ and $\mathrm{supp}(\phi) \subset \overline{B(0,1)}$. Setting $\phi_\epsilon(x) := \frac{1}{\epsilon^d}\phi(\frac{x}{\epsilon})$ for $\epsilon > 0$, it can be shown for $f \in L^p(\Omega)$ with $p \in [1, \infty[$, that $\|f - \phi_\varepsilon * f\|_{L^p} \to 0$ as $\varepsilon \to 0$ and $\|g - \phi_\varepsilon * g\|_{W^{m,p}} \to 0$ as $\varepsilon \to 0$ if $g \in W^{m,p}(\mathbb{R}^d)$. In particular, $f * \phi_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$, which as shown in the lecture, implies that $\mathcal{C}^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ for $p < \infty$ and more generally convolution can be used to find smooth functions with specific properties.

Example 2.1) [Partition of unity]
Let $N \in \mathbb{N}$ and for $i \in \{1, \ldots, N\}$ let $U_i \subset \mathbb{R}^d$ open and $\Omega \subset\subset \bigcup_{i=1}^N U_i$. Show that there exist functions $\xi_i \in \mathcal{C}^\infty(\mathbb{R}^d)$ for $i \in \{1, \ldots, N\}$, such that

$$\begin{cases} 0 \leq \xi_i(x) \leq 1 \qquad \forall x \in \Omega \qquad \forall i \in \{1, \ldots, N\}, \\ \mathrm{supp}(\xi_i) \subset U_i \text{ compact}, \qquad \forall i \in \{1, \ldots, N\}, \\ \sum_{i=1}^N \xi_i(x) = 1 \quad \forall x \in \Omega. \end{cases} \quad (2)$$

In particular, give a rigorous construction and prove all properties you claim they possess.

Hint. *You may use the fact, that there is a mollifier (which can be constructed from* $f(x) = e^{\frac{-1}{1-|x|^2}}\chi_{|x|<1}$*). Also, be aware that division of a smooth function by smooth a function is not necessarily smooth (if the denominator vanishes). Also, be aware of the fact that a* <u>*compact set*</u> $K \subset\subset U$ *(for open* U*) has positive distance to* U^C*, i.e., there is* $\delta > 0$ *such that* $\overline{K + B(0,\delta)} \subset\subset U$*.*

Remark. *A set of functions with property* (2) *is called a partition of unity and is useful as it allows to first consider functions locally in small domains* U_i *before extending to the entirety of* Ω*.*

Example 2.2) [Approximation via convolution]
Let $\Omega \subset \mathbb{R}^d$ be open, and ϕ be a mollifier.

a) Let $f \in W^{m,p}(\Omega)$ for some $m \in \mathbb{N}$, $p < \infty$ and $\epsilon > 0$. Show (without using density) that $\partial^\alpha(f * \phi_\epsilon) = [\partial^\alpha f] * \phi_\epsilon$ on $\Omega_\epsilon := \{x \in \Omega \mid B_\epsilon(x) \subset \Omega\}$ for $|\alpha| \leq m$.

b) Let $f \in \mathcal{C}(\Omega)$ be continuous. Show that $f * \phi_\epsilon \to f$ as $\epsilon \to 0$ pointwise and uniformly on compact sets $K \subset\subset \Omega$.

Remark. *Convolution can be used to find smooth approximations of Sobolev functions. The corresponding proofs are based on the results a) and b).*

Example 2.3) [Poincaré and Lax-Milgram]
Let $\Omega \subset \mathbb{R}^d$ be open and bounded in the first variable, i.e., $\Omega \subset \{x \in \mathbb{R}^d \mid |x_1| < M\}$ for some $M > 0$.

a) Show that there is a constant $c = c(\Omega) > 0$ such that

$$\|u\|_{L^p} \le c\|\nabla u\|_{L^p} \qquad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3}$$

b) Show that given $f \in W^{m,2}(\Omega)$ and $m, n \in \mathbb{N}_0$ with $m < n$, there is a unique $\tilde{f} \in W_0^{n,2}(\Omega)$, such that

$$\sum_{|\alpha|=n} \langle \partial^\alpha \tilde{f}, \partial^\alpha g\rangle_{L^2} = \langle f, g\rangle_{W^{m,2}} \qquad \text{for all } g \in W_0^{n,2}(\Omega). \tag{4}$$

Hint. *Show a) first for smooth functions and use density. Use the Riesz representation theorem for b).*

Remark. *Results of type a) are known as Poincaré inequalities and are quite important in the analysis of partial differential equations as they allow to infer estimates from just controlling the derivatives. The second result is a version of the Lemma of Lax-Milgram used to show unique existence of weak solutions (see lecture on PDEs).*

Example 2.4) [Fundamental lemma of calculus of variations]
Let $\Omega \subset \mathbb{R}^d$ be an open.

a) Show for $f \in L^1(\Omega)$, that

$$\|f\|_{L^1(\Omega)} = \sup\left\{ \int_\Omega f v \,\mathrm{d}x \;\middle|\; v \in \mathcal{C}_c^\infty(\Omega),\ \|v\|_\infty \le 1 \right\}. \tag{5}$$

b) Conclude for $f \in L^1_{\mathrm{loc}}(\Omega)$ that

$$f = 0 \text{ a.e.} \qquad \text{if and only if} \qquad \int_\Omega f(x)v(x)\,\mathrm{d}x = 0 \qquad \text{for all } v \in \mathcal{C}_c^\infty(\Omega). \tag{6}$$

Conclude that weak derivatives are unique.

Hint. *For a) first consider how a function $v \in L^\infty(\Omega)$ would look like such that $\|f\|_{L^1} = \int_\Omega f v \,\mathrm{d}x$ (such exists due to the Hahn-Banach theorem). Use a smoothed version of said function to conclude the identity* (5)*.*

Remark. *Statement* (6) *is known as the fundamental theorem of calculus of variations (Fundamentallemma der Variationsrechnung), and is of utmost importance as it implies that L^1_{loc} functions are uniquely determined by "testing" with very smooth functions. Such considerations are for example important in finding, that weak derivatives are unique or understanding L^1_{loc} as a proper subset of distributions (see later in the lecture).*